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Supersymmetric duality rotations

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Abstract: We derive $\mathcal{N}=1,2$ superfield equations as the conditions for a (nonlinear) theory of one abelian $\mathcal{N}=1$ or $\mathcal{N}=2$ vector multiplet to be duality invariant. The $\mathcal{N}=1$ super Born-Infeld action is a particular solution of the corresponding equation. A family of duality invariant nonlinear $\mathcal{N}=1$ supersymmetric theories is described. We present the solution of the $\mathcal{N}=2$ duality equation which reduces to the $\mathcal{N}=1$ Born-Infeld action when the $(0,1/2)$ part of $\mathcal{N}=2$ vector multiplet is switched off. We also propose a constructive perturbative scheme to compute duality invariant $\mathcal{N}=2$ superconformal actions.

Keywords: Supersymmetry and Duality, Extended Supersymmetry.

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Contents

1. Introduction

The general theory of duality invariance of abelian gauge theory was developed in [1, 2] and further elaborated in a series of publications (see [3]–[10] and references therein). In this paper we generalize the duality equation of Gaillard and Zumino [6, 7], also obtained independently in [4], to $\mathcal{N}=1,2$ supersymmetric theories. This duality equation is the condition for a theory with lagrangian $L(F_{ab})$ to be invariant under U(1) duality transformations

$$\delta F = \lambda\, G\,, \qquad \delta G = -\lambda\, F\,, \tag{1.1}$$

where

$$\tilde{G}_{ab} = \frac{1}{2}\,\varepsilon_{abcd}\,G^{cd} = 2\,\frac{\partial L}{\partial F^{ab}}\,. \tag{1.2}$$

The equation reads

$$G^{ab}\,\tilde{G}_{ab} + F^{ab}\,\tilde{F}_{ab} = 0 \tag{1.3}$$

and presents a nontrivial constraint on the lagrangian.

The Born-Infeld (BI) theory [11] is a particular solution of eq. (1.3). The BI action naturally appears in string theory [12, 13] (see [14] for a recent review). Its $\mathcal{N}=1$ supersymmetric generalization [16] (see also [15]) turns out to be the action for a Goldstone multiplet associated with partial breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ supersymmetry [17, 18]. It has been conjectured [19] that a $\mathcal{N}=2$ supersymmetric generalization of the BI action should provide a model for partial breakdown $\mathcal{N}=4\to\mathcal{N}=2$, with the $\mathcal{N}=2$ vector multiplet being the corresponding Goldstone field, but the existing mechanisms of partial supersymmetry breaking are very difficult to implement in the $\mathcal{N}=4$ case. A candidate for $\mathcal{N}=2$ BI action has been suggested in [20]. It correctly reduces to the Cecotti-Ferrara action [16] once the $(0,1/2)$ part of the $\mathcal{N}=2$ vector multiplet is switched off. However, there exist infinitely many $\mathcal{N}=2$ superfield actions with that property. Therefore, requiring the correct $\mathcal{N}=1$ reduction does not suffice to fix a proper $\mathcal{N}=2$ generalization of the BI action. One has to impose additional physical requirements. Since no mechanism for partial $\mathcal{N}=4 \to \mathcal{N}=2$ breaking is currently available, it is natural to look for the $\mathcal{N}=2$ BI action as a solution of the supersymmetric generalization of the Gaillard-Zumino equation (1.3).

In this paper we find $\mathcal{N}=1,\ 2$ supersymmetric generalizations of the duality equation (1.3). They are presented in eqs. (2.8) and (3.10), respectively. It is not surprising that the Cecotti-Ferrara action [16] is a solution of the $\mathcal{N}=1$ duality equation. In contrast, the action proposed in [20] does not satisfy the $\mathcal{N}=2$ duality equation. However, the key to the construction of duality invariant $\mathcal{N}=2$ BI action was given in [21] where a nonlinear $\mathcal{N}=2$ superfield constraint was introduced as a minimal extension of that generating the $\mathcal{N}=1$ BI action [17, 18]. It was asserted that the constrained superfield introduced does generate the $\mathcal{N}=2$ action given in [20]. While this claim is incorrect, the constrained superfield nevertheless does generate the duality invariant $\mathcal{N}=2$ action that reduces to the $\mathcal{N}=1$ BI action after the $(0,1/2)$ part of the $\mathcal{N}=2$ vector multiplet is switched off.

One application of the $\mathcal{N}=2$ duality equation may be to compute the duality invariant low-energy effective actions of supersymmetric gauge theories. The $\mathcal{N}=4$ super Yang-Mills theory is expected to be self-dual [22, 23]. It was proposed in [24] to look for its low-energy action on the Coulomb branch as a solution of the self-duality equation via the $\mathcal{N}=2$ superfield Legendre transformation, and a few subleading corrections to the low-energy action were determined. For non-supersymmetric theories it was shown in [7] that the Gaillard-Zumino equation (1.3) implies self-duality via Legendre transformation. The Gaillard-Zumino equation is much simpler to solve and this advantage becomes essential in supersymmetric theories, where the procedure of inverting the Legendre transformation appears to be more involved at higher orders of perturbation theory [24].

We have already remarked that (1.3) implies self-duality via Legendre transformation, but it is in fact a stronger condition. With reference to recent interest in the (supersymmetric) BI action within the context of D-branes, this stronger condition is in fact what one would like to impose. As was noted in [25, 26, 27], the D3-brane world-volume action, which contains, in addition to the gauge field also the axion and the dilaton fields, possesses a non-trivial $\mathrm{SL}(2,\mathbb{R})$ symmetry. The BI action we are considering corresponds to the CP-even part of this action for the special choice of vanishing axion and dilaton. This background is invariant precisely under the $\mathrm{U}(1) \subset \mathrm{SL}(2,\mathbb{R})$ duality group we are considering.

JHEP03(2000)034

Our paper is organized as follows. In section 2 we derive the $\mathcal{N}=1$ generalization of the Gaillard-Zumino equation and give a family of duality invariant nonlinear $\mathcal{N}=1$ models. The $\mathcal{N}=1$ BI action [16] is a special member of this family. We also introduce a superconformally invariant generalization of the $\mathcal{N}=1$ BI action by coupling the vector multiplet to a scalar multiplet. In section 3 we present the $\mathcal{N}=2$ duality equation and derive its nonperturbative solution that reduces to the $\mathcal{N}=1$ BI action when the $(0,1/2)$ part of $\mathcal{N}=2$ vector multiplet is switched off. We also develop a consistent perturbative scheme of computing duality invariant $\mathcal{N}=2$ superconformal actions. In appendix A we discuss the general structure of the duality equation in the non-supersymmetric case and we show that any solution of (1.3) admits a supersymmetric extension. In appendix B we give an explicit proof that the $\mathcal{N}=2$ BI action is self-dual with respect to Legendre transformation.

2. $\mathcal{N}=1$ duality rotations

Let $S[W,\bar{W}]$ be the action describing the dynamics of a single $\mathcal{N}=1$ vector multiplet. The (anti) chiral superfield strengths $\bar{W}_{\dot\alpha}$ and W_α,[1]

$$W_\alpha = -\frac{1}{4}\,\bar{D}^2 D_\alpha V\,, \qquad \bar{W}_{\dot\alpha} = -\frac{1}{4}\,D^2 \bar{D}_{\dot\alpha} V\,, \tag{2.1}$$

are defined in terms of a real unconstrained prepotential V. As a consequence, the strengths are constrained superfields, that is they satisfy the Bianchi identity

$$D^\alpha W_\alpha = \bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha}\,. \tag{2.2}$$

Suppose that $S[W,\bar{W}]$ can be unambiguously defined[2] as a functional of *unconstrained* (anti) chiral superfields $\bar{W}_{\dot\alpha}$ and W_α. Then, one can define (anti) chiral superfields $\bar{M}_{\dot\alpha}$ and M_α as

$$\mathrm{i}\,M_\alpha \equiv 2\,\frac{\delta}{\delta W^\alpha}\,S[W,\bar{W}]\,, \qquad -\mathrm{i}\,\bar{M}^{\dot\alpha} \equiv 2\,\frac{\delta}{\delta \bar{W}_{\dot\alpha}}\,S[W,\bar{W}]\,. \tag{2.3}$$

The equation of motion following from the action $S[W,\bar{W}]$ reads

$$D^\alpha M_\alpha = \bar{D}_{\dot\alpha}\bar{M}^{\dot\alpha}\,. \tag{2.4}$$

Since the Bianchi identity (2.2) and the equation of motion (2.4) have the same functional form, one may consider infinitesimal U(1) duality transformations

$$\delta W_\alpha = \lambda\,M_\alpha\,, \qquad \delta M_\alpha = -\lambda\,W_\alpha\,. \tag{2.5}$$

[1] Our $\mathcal{N}=1$ conventions correspond to [28].

[2] This is always possible if $S[W,\bar{W}]$ does not involve the combination $D^\alpha W_\alpha$ as an independent variable.

To preserve the definition (2.3) of M_α and its conjugate, the action should transform as

$$\delta S = -\frac{\mathrm{i}}{4}\lambda \int \mathrm{d}^6 z \, \{W^\alpha W_\alpha - M^\alpha M_\alpha\} + \frac{\mathrm{i}}{4}\lambda \int \mathrm{d}^6 \bar{z} \, \{\bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} - \bar{M}_{\dot{\alpha}} \bar{M}^{\dot{\alpha}}\} , \qquad (2.6)$$

in complete analogy with the analysis of [7] for the non-supersymmetric case.[3] On the other hand, S is a functional of W_α and $\bar{W}_{\dot{\alpha}}$ only, and therefore its variation under (2.5) is

$$\delta S = \frac{\mathrm{i}}{2}\lambda \int \mathrm{d}^6 z \, M^\alpha M_\alpha - \frac{\mathrm{i}}{2}\lambda \int \mathrm{d}^6 \bar{z} \, \bar{M}_{\dot{\alpha}} \bar{M}^{\dot{\alpha}} . \qquad (2.7)$$

Since these two variations must coincide, we arrive at the following reality condition

$$\int \mathrm{d}^6 z \, \Big(W^\alpha W_\alpha + M^\alpha M_\alpha\Big) = \int \mathrm{d}^6 \bar{z} \, \Big(\bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} + \bar{M}_{\dot{\alpha}} \bar{M}^{\dot{\alpha}}\Big) . \qquad (2.8)$$

In eq. (2.8), the superfields M_α and $\bar{M}_{\dot{\alpha}}$ are defined as in (2.3), and W_α and $\bar{W}_{\dot{\alpha}}$ should be considered as *unconstrained* chiral and antichiral superfields, respectively. Eq. (2.8) is the condition for the $\mathcal{N}=1$ supersymmetric theory to be duality invariant. We call it the $\mathcal{N}=1$ duality equation.

A nontrivial solution of eq. (2.8) is the $\mathcal{N} = 1$ supersymmetric Born-Infeld action [16, 17, 18] (see also [15])

$$S_{\mathrm{BI}} = \frac{1}{4}\int \mathrm{d}^6 z \, W^2 + \frac{1}{4}\int \mathrm{d}^6 \bar{z} \, \bar{W}^2 + \frac{1}{g^4}\int \mathrm{d}^8 z \, \frac{W^2 \, \bar{W}^2}{1 + \frac{1}{2} A + \sqrt{1 + A + \frac{1}{4} B^2}} ,$$
$$A = \frac{1}{2g^4}\Big(D^2 \, W^2 + \bar{D}^2 \, \bar{W}^2\Big) , \qquad B = \frac{1}{2g^4}\Big(D^2 \, W^2 - \bar{D}^2 \, \bar{W}^2\Big) , \qquad (2.9)$$

where g is a coupling constant. This is a model for a Goldstone multiplet associated with partial breaking of $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry [17, 18] (see also [14]), with W_α being the Goldstone multiplet.

New examples of $\mathcal{N}=1$ duality invariant models can be obtained by considering a general action of the form (see also appendix A)

$$S = \frac{1}{4}\int \mathrm{d}^6 z \, W^2 + \frac{1}{4}\int \mathrm{d}^6 \bar{z} \, \bar{W}^2 + \frac{1}{2}\int \mathrm{d}^8 z \, W^2 \, \bar{W}^2 \, L(D^2 \, W^2, \bar{D}^2 \, \bar{W}^2) , \qquad (2.10)$$

where $L(u, \bar{u})$ is a real analytic function of the complex variable $u \equiv D^2 \, W^2$ and its conjugate. One finds

$$\mathrm{i} \, M_\alpha = W_\alpha \left\{ 1 - \frac{1}{2} \bar{D}^2 \left[\bar{W}^2 \left(L + D^2 \Big(W^2 \frac{\partial L}{\partial u} \Big) \right) \right] \right\} . \qquad (2.11)$$

[3] Note that the action S itself is not duality invariant, but rather the combination $S - \frac{\mathrm{i}}{4}\int \mathrm{d}^6 z \, W M + \frac{\mathrm{i}}{4}\int \mathrm{d}^6 \bar{z} \, \bar{W} \bar{M}$. The invariance of this functional under a finite U(1) duality rotation by $\pi/2$, is equivalent to the self-duality of S under Legendre transformation, $S[W, \bar{W}] - \frac{\mathrm{i}}{2}\int \mathrm{d}^6 z \, W W_{\mathrm{D}} + \frac{\mathrm{i}}{2}\int \mathrm{d}^6 \bar{z} \, \bar{W} \bar{W}_{\mathrm{D}} = S[W_{\mathrm{D}}, \bar{W}_{\mathrm{D}}]$, with W_{D} being the dual chiral field strength.

Then, eq. (2.8) leads to

$$4 \int \mathrm{d}^8 z\, W^2\, \bar{W}^2\, \left(\Gamma - \bar{\Gamma}\right) = \int \mathrm{d}^8 z\, W^2\, \bar{W}^2\, \left(\Gamma^2\, \bar{D}^2\, \bar{W}^2 - \bar{\Gamma}^2\, D^2\, W^2\right), \qquad (2.12)$$

where

$$\Gamma \equiv L + \frac{\partial L}{\partial u}\, D^2\, W^2 = \frac{\partial (u\, L)}{\partial u}\,. \qquad (2.13)$$

Since the latter functional relation must be satisfied for arbitrary (anti) chiral superfields $\bar{W}_{\dot{\alpha}}$ and W_α, we arrive at the following differential equation for $L(u, \bar{u})$:

$$4 \left(\frac{\partial (u\, L)}{\partial u} - \frac{\partial (\bar{u}\, L)}{\partial \bar{u}} \right) = \bar{u} \left(\frac{\partial (u\, L)}{\partial u} \right)^2 - u \left(\frac{\partial (\bar{u}\, L)}{\partial \bar{u}} \right)^2. \qquad (2.14)$$

Similar to the non-supersymmetric case [4, 7], the general solution of this equation involves an arbitrary real analytic function of a single real argument, $f(\bar{u}\, u)$.[4] It is an easy exercise to check that the $\mathcal{N} = 1$ BI action (2.9) satisfies eq. (2.14).

We conclude this section by giving an extension of the model (2.9), in which the vector multiplet is coupled to an external chiral superfield Φ in such a way that the system is not only duality invariant but also invariant under the $\mathcal{N} = 1$ superconformal group. The action is

$$S = \frac{1}{4} \int \mathrm{d}^6 z\, W^2 + \frac{1}{4} \int \mathrm{d}^6 \bar{z}\, \bar{W}^2 + \int \mathrm{d}^8 z\, \frac{W^2\, \bar{W}^2\, (\Phi\, \bar{\Phi})^{-2}}{1 + \frac{1}{2}\mathbf{A} + \sqrt{1 + \mathbf{A} + \frac{1}{4}\mathbf{B}^2}}\,, \qquad (2.15)$$

$$\mathbf{A} = \frac{1}{2} \left(\frac{D^2}{\bar{\Phi}^2} \left(\frac{W^2}{\Phi^2} \right) + \frac{\bar{D}^2}{\Phi^2} \left(\frac{\bar{W}^2}{\bar{\Phi}^2} \right) \right), \qquad \mathbf{B} = \frac{1}{2} \left(\frac{D^2}{\bar{\Phi}^2} \left(\frac{W^2}{\Phi^2} \right) - \frac{\bar{D}^2}{\Phi^2} \left(\frac{\bar{W}^2}{\bar{\Phi}^2} \right) \right).$$

Superconformal invariance follows from the superconformal transformation properties as given in [30]. The theory is invariant under the duality rotations (2.5) with Φ being inert. By its very construction, the action is also invariant under global phase transformations of Φ. In a sense, this model is analogous to the BI theory coupled to dilaton and axion fields [5, 8].

Similar to the analysis of [17, 18], it is possible to show that the action (2.15) can be represented in the form

$$S = \frac{1}{4} \int \mathrm{d}^6 z\, \mathbf{X} + \frac{1}{4} \int \mathrm{d}^6 \bar{z}\, \bar{\mathbf{X}}\,, \qquad (2.16)$$

where the chiral superfield $\mathbf{X}$ is a functional of W_α and $\bar{W}_{\dot{\alpha}}$ such that it satisfies the nonlinear constraint

$$\mathbf{X} + \mathbf{X}\, \frac{\bar{D}^2}{4\Phi^2} \left(\frac{\bar{\mathbf{X}}}{\bar{\Phi}^2} \right) = W^2\,. \qquad (2.17)$$

The $\mathcal{N} = 1$ BI theory is obtained from this model by freezing Φ.

[4] Among non-supersymmetric duality invariant models, only the Maxwell action and the BI action satisfy the requirement of shock-free wave propagation [29].

JHEP03(2000)034

More generally, for any duality invariant system defined by eqs. (2.10) and (2.14), the replacement

$$W^2\,\bar W^2 \longrightarrow \frac{W^2\,\bar W^2}{\Phi^2\,\bar\Phi^2}\,, \qquad D^2 \longrightarrow \frac{1}{\bar\Phi^2}\,D^2\,\frac{1}{\Phi^2} \tag{2.18}$$

in (2.10) preserves the duality invariance but turns the action into a $\mathcal{N}=1$ superconformal functional.

3. $\mathcal{N}=2$ duality rotations

We now generalize the results of the previous section to the case of $\mathcal{N}=2$ supersymmetry. We will work in $\mathcal{N}=2$ global superspace $\mathbb{R}^{4|8}$ parametrized by $\mathcal{Z}^A=(x^a,\theta^\alpha_i,\bar\theta^i_{\dot\alpha})$, where $i=\underline{1},\underline{2}$. The flat covariant derivatives $\mathcal{D}_A=(\partial_a,\mathcal{D}^i_\alpha,\bar{\mathcal{D}}^{\dot\alpha}_i)$ satisfy the standard algebra

$$\{\mathcal{D}^i_\alpha,\mathcal{D}^j_\beta\}=\{\bar{\mathcal{D}}_{\dot\alpha i},\bar{\mathcal{D}}_{\dot\beta j}\}=0\,, \qquad \{\mathcal{D}^i_\alpha,\bar{\mathcal{D}}_{\dot\alpha j}\}=-2\,\mathrm{i}\,\delta^i_j\,(\sigma^a)_{\alpha\dot\alpha}\,\partial_a\,. \tag{3.1}$$

Throughout this section, we will use the notation:

$$\begin{aligned}
\mathcal{D}^{ij} &\equiv \mathcal{D}^{\alpha(i}\mathcal{D}^{j)}_\alpha=\mathcal{D}^{\alpha i}\mathcal{D}^j_\alpha\,, &\qquad \bar{\mathcal{D}}^{ij} &\equiv \bar{\mathcal{D}}^{(i}_{\dot\alpha}\bar{\mathcal{D}}^{j)\,\dot\alpha}=\bar{\mathcal{D}}^i_{\dot\alpha}\bar{\mathcal{D}}^{j\,\dot\alpha}\\
\mathcal{D}^4 &\equiv \frac{1}{16}(\mathcal{D}^{\underline{1}})^2\,(\mathcal{D}^{\underline{2}})^2\,, & \bar{\mathcal{D}}^4 &\equiv \frac{1}{16}(\bar{\mathcal{D}}_{\underline{1}})^2\,(\bar{\mathcal{D}}_{\underline{2}})^2\,.
\end{aligned} \tag{3.2}$$

An integral over the full superspace can be reduce to one over the chiral subspace or over the antichiral subspace as follows:

$$\int \mathrm{d}^{12}\mathcal{Z}\;\mathcal{L}(\mathcal{Z})=\int \mathrm{d}^8\mathcal{Z}\;\mathcal{D}^4\mathcal{L}(\mathcal{Z})=\int \mathrm{d}^8\bar{\mathcal{Z}}\;\bar{\mathcal{D}}^4\mathcal{L}(\mathcal{Z})\,. \tag{3.3}$$

3.1 $\mathcal{N}=2$ duality equation

The discussion in this subsection is completely analogous to the one presented in the first part of section 2. We will thus be brief. If $\mathcal{S}[\mathcal{W},\bar{\mathcal{W}}]$ is the action describing the dynamics of a single $\mathcal{N}=2$ vector multiplet, the (anti) chiral superfield strengths $\bar{\mathcal{W}}$ and $\mathcal{W}$ are [31]

$$\mathcal{W}=\bar{\mathcal{D}}^4\mathcal{D}^{ij}\,V_{ij}\,, \qquad \bar{\mathcal{W}}=\mathcal{D}^4\bar{\mathcal{D}}^{ij}\,V_{ij} \tag{3.4}$$

in terms of a real unconstrained prepotential $V_{(ij)}$. The strengths then satisfy the Bianchi identity [32]

$$\mathcal{D}^{ij}\,\mathcal{W}=\bar{\mathcal{D}}^{ij}\,\bar{\mathcal{W}}\,. \tag{3.5}$$

Suppose that $\mathcal{S}[\mathcal{W},\bar{\mathcal{W}}]$ can be unambiguously defined as a functional of *unconstrained* (anti) chiral superfields $\bar{\mathcal{W}}$ and $\mathcal{W}$. Then, one can define (anti) chiral superfields $\bar{\mathcal{M}}$ and $\mathcal{M}$ as

$$\mathrm{i}\,\mathcal{M}\equiv 4\,\frac{\delta}{\delta\mathcal{W}}\,\mathcal{S}[\mathcal{W},\bar{\mathcal{W}}]\,, \qquad -\mathrm{i}\,\bar{\mathcal{M}}\equiv 4\,\frac{\delta}{\delta\bar{\mathcal{W}}}\,\mathcal{S}[\mathcal{W},\bar{\mathcal{W}}] \tag{3.6}$$

JHEP03(2000)034

in terms of which the equations of motion read

$$\mathcal{D}^{ij}\,\mathcal{M} = \bar{\mathcal{D}}^{ij}\,\bar{\mathcal{M}}\,. \tag{3.7}$$

Again, since the Bianchi identity (3.5) and the equation of motion (3.7) have the same functional form, one can consider infinitesimal U(1) duality transformations

$$\delta\mathcal{W} = \lambda\,\mathcal{M}\,, \qquad \delta\mathcal{M} = -\lambda\,\mathcal{W}\,. \tag{3.8}$$

Repeating the analysis of Gaillard and Zumino [7] (see also section 2), we now have to impose

$$\begin{aligned}\delta\mathcal{S} &= -\frac{\mathrm{i}}{8}\lambda\int \mathrm{d}^8\mathcal{Z}\left(\mathcal{W}^2 - \mathcal{M}^2\right) + \frac{\mathrm{i}}{8}\lambda\int \mathrm{d}^8\bar{\mathcal{Z}}\left(\bar{\mathcal{W}}^2 - \bar{\mathcal{M}}^2\right) \\ &= \frac{\mathrm{i}}{4}\lambda\int \mathrm{d}^8\mathcal{Z}\,\mathcal{M}^2 - \frac{\mathrm{i}}{4}\lambda\int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{M}}^2\,.\end{aligned} \tag{3.9}$$

The theory is thus duality invariant provided the following reality condition is satisfied:

$$\int \mathrm{d}^8\mathcal{Z}\left(\mathcal{W}^2 + \mathcal{M}^2\right) = \int \mathrm{d}^8\bar{\mathcal{Z}}\left(\bar{\mathcal{W}}^2 + \bar{\mathcal{M}}^2\right). \tag{3.10}$$

Here $\mathcal{M}$ and $\bar{\mathcal{M}}$ are defined as in (3.6), and $\mathcal{W}$ and $\bar{\mathcal{W}}$ should be considered as *unconstrained* chiral and antichiral superfields, respectively. Eq. (3.10) serves as our master functional equation to determine duality invariant models of the $\mathcal{N}=2$ vector multiplet. We remark that, as in the $\mathcal{N}=1$ case, the action itself is not duality invariant, but

$$\delta\left(\mathcal{S} - \frac{\mathrm{i}}{8}\int \mathrm{d}^8\mathcal{Z}\,\mathcal{M}\mathcal{W} + \frac{\mathrm{i}}{8}\int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{M}}\bar{\mathcal{W}}\right) = 0\,. \tag{3.11}$$

The invariance of the latter functional under a finite U(1) duality rotation by $\pi/2$, is equivalent to the self-duality of $\mathcal{S}$ under Legendre transformation,

$$\mathcal{S}[\mathcal{W},\bar{\mathcal{W}}] - \frac{\mathrm{i}}{4}\int \mathrm{d}^8\mathcal{Z}\,\mathcal{W}\mathcal{W}_{\mathrm{D}} + \frac{\mathrm{i}}{4}\int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{W}}\bar{\mathcal{W}}_{\mathrm{D}} = \mathcal{S}[\mathcal{W}_{\mathrm{D}},\bar{\mathcal{W}}_{\mathrm{D}}]\,, \tag{3.12}$$

where the dual chiral field strength $\mathcal{W}_{\mathrm{D}}$ is given by eq. (B.2).

3.2 $\mathcal{N}=2$ BI action

Recently, Ketov [20] suggested the following action

$$\begin{aligned}\mathcal{S}_{\mathrm{BI}} &= \frac{1}{8}\int \mathrm{d}^8\mathcal{Z}\,\mathcal{W}^2 + \frac{1}{8}\int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{W}}^2 + \frac{1}{4}\int \mathrm{d}^{12}\mathcal{Z}\,\frac{\mathcal{W}^2\,\bar{\mathcal{W}}^2}{1 - \frac{1}{2}\mathcal{A} + \sqrt{1 - \mathcal{A} + \frac{1}{4}\mathcal{B}^2}}\,, \\ \mathcal{A} &= \mathcal{D}^4\,\mathcal{W}^2 + \bar{\mathcal{D}}^2\,\bar{\mathcal{W}}^2\,, \qquad \mathcal{B} = \mathcal{D}^4\,\mathcal{W}^2 - \bar{\mathcal{D}}^4\,\bar{\mathcal{W}}^2\end{aligned} \tag{3.13}$$

as the $\mathcal{N}=2$ supersymmetric generalization of the BI action. We will first demonstrate that it indeed reduces to the $\mathcal{N}=1$ BI action. We then show that this condition is not strong enough to uniquely fix the $\mathcal{N}=2$ BI action but this is possible if, in addition, one imposes eq. (3.10).

Given a $\mathcal{N}=2$ superfield U, its $\mathcal{N}=1$ projection is defined to be $U| = \mathrm{U}(\mathcal{Z})|_{\theta_{\underline{2}}=\bar{\theta}^{\underline{2}}=0}$. The $\mathcal{N}=2$ vector multiplet contains two independent chiral $\mathcal{N}=1$ components

$$\mathcal{W}| = \sqrt{2}\,\Phi\,, \qquad \mathcal{D}^{\underline{2}}_{\alpha}\mathcal{W}| = 2\mathrm{i}\,W_\alpha\,, \qquad (\mathcal{D}^{\underline{2}})^2\mathcal{W}| = \sqrt{2}\,\bar{D}^2\bar{\Phi}\,. \tag{3.14}$$

Using in addition that

$$\int \mathrm{d}^8\mathcal{Z} = -\frac{1}{4}\int \mathrm{d}^6 z\,(\mathcal{D}^{\underline{2}})^2\,, \qquad \int \mathrm{d}^{12}\mathcal{Z} = \frac{1}{16}\int \mathrm{d}^8 z\,(\mathcal{D}^{\underline{2}})^2\,(\bar{\mathcal{D}}_{\underline{2}})^2\,, \tag{3.15}$$

the free $\mathcal{N}=2$ vector multiplet action straightforwardly reduces to $\mathcal{N}=1$ superfields

$$\mathcal{S}_{\text{free}} = \frac{1}{8}\int \mathrm{d}^8\mathcal{Z}\,\mathcal{W}^2 + \frac{1}{8}\int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{W}}^2 = \int \mathrm{d}^8 z\,\bar{\Phi}\Phi + \frac{1}{4}\int \mathrm{d}^6 z\,W^2 + \frac{1}{4}\int \mathrm{d}^6\bar{z}\,\bar{W}^2\,. \tag{3.16}$$

If one switches off Φ,

$$\Phi = 0 \qquad \Longrightarrow \qquad (\mathcal{D}^{\underline{2}})^2\mathcal{W}| = 0\,, \tag{3.17}$$

the action (3.13) reduces to the $\mathcal{N}=1$ BI theory (2.9) (with $g=1$). However, as we will now demonstrate, there exist infinitely many $\mathcal{N}=2$ actions with that property.[5] To demonstrate why this is possible, consider the following obviously different functionals

$$\int \mathrm{d}^{12}\mathcal{Z}\,\mathcal{W}^2\bar{\mathcal{W}}^2\,\{\,(\mathcal{D}^4\mathcal{W}^2)^2\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2 + (\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2)^2\mathcal{D}^4\mathcal{W}^2\,\}\,,$$
$$\int \mathrm{d}^{12}\mathcal{Z}\,\mathcal{W}^2\bar{\mathcal{W}}^2\,\Big\{\,(\mathcal{D}^4\mathcal{W}^2)\,\bar{\mathcal{D}}^4\Big[\bar{\mathcal{W}}^2\mathcal{D}^4\mathcal{W}^2\Big] + (\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2)\,\mathcal{D}^4\Big[\mathcal{W}^2\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2\Big]\Big\}\,.$$

They coincide under (3.17). Therefore, the requirement of correct $\mathcal{N}=1$ reduction is too weak to fix a proper $\mathcal{N}=2$ generalization of the BI action.[6]

We suggest to search for a $\mathcal{N}=2$ generalization of the BI action as a solution of the $\mathcal{N}=2$ duality equation (3.10) compatible with the requirement to give the correct $\mathcal{N}=1$ reduction. We have checked to some order in perturbation theory

[5]The property $W_\alpha\,W_\beta\,W_\gamma = 0$ of the $\mathcal{N}=1$ vector multiplet, which is crucial in the discussion of the $\mathcal{N}=1$ BI action, has no direct analog for its $\mathcal{N}=2$ counterpart.

[6]It was claimed in [20, 21] that the action (3.13) is self-dual with respect to the $\mathcal{N}=2$ Legendre transformation. This is, however, not correct.

that these two requirements uniquely fix the solution:

$$
\begin{aligned}
\mathcal{S}_{\rm BI} &= \frac{1}{8}\int {\rm d}^8\mathcal{Z}\mathcal{W}^2 + \frac{1}{8}\int {\rm d}^8\bar{\mathcal{Z}}\bar{\mathcal{W}}^2 + \mathcal{S}_{\rm int}\,,\\
\mathcal{S}_{\rm int} &= \frac{1}{8}\int {\rm d}^{12}\mathcal{Z}\mathcal{W}^2\bar{\mathcal{W}}^2\Big\{1 + \frac{1}{2}\Big(\mathcal{D}^4\mathcal{W}^2 + \bar{\mathcal{D}}^4\bar{\mathcal{W}}^2\Big) + \frac{1}{4}\Big((\mathcal{D}^4\mathcal{W}^2)^2 + (\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2)^2\Big) +\\
&\qquad + \frac{3}{4}(\mathcal{D}^4\mathcal{W}^2)(\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2) + \frac{1}{8}\Big((\mathcal{D}^4\mathcal{W}^2)^3 + (\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2)^3\Big) +\\
&\qquad + \frac{1}{2}\Big((\mathcal{D}^4\mathcal{W}^2)^2(\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2) + (\mathcal{D}^4\mathcal{W}^2)(\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2)^2\Big) +\\
&\qquad + \frac{1}{4}\Big((\mathcal{D}^4\mathcal{W}^2)\bar{\mathcal{D}}^4\Big[\bar{\mathcal{W}}^2\mathcal{D}^4\mathcal{W}^2\Big] + (\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2)\mathcal{D}^4\Big[\mathcal{W}^2\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2\Big]\Big)\Big\} +\\
&\qquad + O(\mathcal{W}^{12})\,. \qquad (3.18)
\end{aligned}
$$

The expression in the last two lines of (3.18) constitutes the leading perturbative corrections where our solution of the duality equation (3.10) differs from the action (3.13).

We now present the nonperturbative solution of (3.10) which reduces to the $\mathcal{N}=1$ BI action (2.9) under the condition (3.17). The action reads

$$\mathcal{S}_{\rm BI} = \frac{1}{4}\int {\rm d}^8\mathcal{Z}\,\mathcal{X} + \frac{1}{4}\int {\rm d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{X}}\,, \qquad (3.19)$$

where the chiral superfield $\mathcal{X}$ is a functional of $\mathcal{W}$ and $\bar{\mathcal{W}}$ defined via the constraint[7]

$$\mathcal{X} = \mathcal{X}\,\bar{\mathcal{D}}^4\bar{\mathcal{X}} + \frac{1}{2}\mathcal{W}^2\,. \qquad (3.20)$$

Solving it iteratively for $\mathcal{X}$ one may verify the equivalence of (3.19) and (3.18) up to the indicated order. The constraint (3.20) was introduced in [21] as a $\mathcal{N}=2$ generalization of that generating the $\mathcal{N}=1$ BI action (2.9) [17, 18] (see eq. (2.17)). It was also claimed in [21] that the action (3.13) can be equivalently described by eqs. (3.19) and (3.20). This is clearly incorrect, since they lead to the action (3.18) rather than to (3.13). But the constraint (3.20) has a deep origin: the $\mathrm{SL}(2,\mathbb{R})$ invariant system introduced in [8] admits a minimal $\mathcal{N}=2$ extension on the base of the constraint (3.20) such that the original $\mathrm{SL}(2,\mathbb{R})$ invariance remains intact.

Let us prove that the system described by eqs. (3.19) and (3.20) provides a solution of the duality equation (3.10). Under an infinitesimal variation of $\mathcal{W}$ only, we have

$$
\begin{aligned}
\delta_{\mathcal{W}}\mathcal{X} &= \delta_{\mathcal{W}}\mathcal{X}\,\bar{\mathcal{D}}^4\bar{\mathcal{X}} + \mathcal{X}\,\bar{\mathcal{D}}^4\delta_{\mathcal{W}}\bar{\mathcal{X}} + \mathcal{W}\,\delta\mathcal{W}\,,\\
\delta_{\mathcal{W}}\bar{\mathcal{X}} &= \delta_{\mathcal{W}}\bar{\mathcal{X}}\,\mathcal{D}^4\mathcal{X} + \bar{\mathcal{X}}\,\mathcal{D}^4\delta_{\mathcal{W}}\mathcal{X}\,. \qquad (3.21)
\end{aligned}
$$

[7]The property $\mathbf{X}^2=0$ of the $\mathcal{N}=1$ constraint (2.17) has no direct analog for $\mathcal{X}$.

From these relations one gets

$$\delta_{\mathcal{W}}\mathcal{X} = \frac{1}{1-\mathcal{Q}}\left[\frac{\mathcal{W}\,\delta\mathcal{W}}{1-\bar{\mathcal{D}}^4\bar{\mathcal{X}}}\right], \qquad \delta_{\mathcal{W}}\bar{\mathcal{X}} = \frac{\bar{\mathcal{X}}}{1-\mathcal{D}^4\mathcal{X}}\,\mathcal{D}^4\delta_{\mathcal{W}}\mathcal{X}\,, \tag{3.22}$$

where

$$\begin{aligned} &\mathcal{Q} = \mathcal{P}\,\bar{\mathcal{P}}\,, && \bar{\mathcal{Q}} = \bar{\mathcal{P}}\,\mathcal{P}\,, \\ &\mathcal{P} = \tfrac{\mathcal{X}}{1-\bar{\mathcal{D}}^4\bar{\mathcal{X}}}\,\bar{\mathcal{D}}^4\,, && \bar{\mathcal{P}} = \tfrac{\bar{\mathcal{X}}}{1-\mathcal{D}^4\mathcal{X}}\,\mathcal{D}^4\,. \end{aligned} \tag{3.23}$$

With these results, it is easy to compute $\mathcal{M}$:

$$\mathrm{i}\,\mathcal{M} = \frac{\mathcal{W}}{1-\bar{\mathcal{D}}^4\bar{\mathcal{X}}}\left\{1+\bar{\mathcal{D}}^4\,\bar{\mathcal{P}}\,\frac{1}{1-\mathcal{Q}}\,\frac{\mathcal{X}}{1-\bar{\mathcal{D}}^4\bar{\mathcal{X}}}+\bar{\mathcal{D}}^4\,\frac{1}{1-\bar{\mathcal{Q}}}\,\frac{\bar{\mathcal{X}}}{1-\mathcal{D}^4\mathcal{X}}\right\}. \tag{3.24}$$

Now, a short calculation gives

$$\mathrm{Im}\int \mathrm{d}^8\mathcal{Z}\left\{\mathcal{M}^2+2\,\frac{1}{1-\mathcal{Q}}\,\frac{\mathcal{X}}{1-\bar{\mathcal{D}}^4\bar{\mathcal{X}}}\right\} = 0\,. \tag{3.25}$$

On the other hand, the constraint (3.20) implies

$$\int \mathrm{d}^8\mathcal{Z}\,\mathcal{X} - \int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{X}} = \frac{1}{2}\int \mathrm{d}^8\mathcal{Z}\,\mathcal{W}^2 - \frac{1}{2}\int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{W}}^2\,, \tag{3.26}$$

and hence

$$\frac{\delta}{\delta\mathcal{W}}\left\{\int \mathrm{d}^8\mathcal{Z}\,\mathcal{X} - \int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{X}}\right\} = \mathcal{W}\,. \tag{3.27}$$

The latter relation can be shown to be equivalent to

$$\frac{1}{1-\mathcal{Q}}\,\frac{\mathcal{X}}{1-\bar{\mathcal{D}}^4\bar{\mathcal{X}}} = \mathcal{P}\,\frac{1}{1-\bar{\mathcal{Q}}}\,\frac{\bar{\mathcal{X}}}{1-\mathcal{D}^4\mathcal{X}} + \mathcal{X}\,. \tag{3.28}$$

Using this result in eq. (3.25), we arrive at the relation

$$\int \mathrm{d}^8\mathcal{Z}\,\mathcal{M}^2 - \int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{M}}^2 = -2\int \mathrm{d}^8\mathcal{Z}\,\mathcal{X} + 2\int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{X}}\,, \tag{3.29}$$

which is equivalent, due to (3.26), to (3.10).

In appendix B we prove the self-duality of the $\mathcal{N}=2$ BI action under Legendre transformation explicitly, although this property already follows from the general analysis of [7] or our discussion in section 3.1.

3.3 Duality invariant $\mathcal{N}=2$ superconformal actions

The $\mathcal{N}=4$ super Yang-Mills theory is believed to be self-dual [22, 23]. It was therefore suggested in [24] to look for its low-energy effective action on the Coulomb branch as a solution to the self-duality equation via the $\mathcal{N}=2$ Legendre transformation such that the leading (second- and fourth- order) terms in the momentum expansion of the action look like

$$\mathcal{S}_{\rm lead} = \frac{1}{8}\int {\rm d}^8\mathcal{Z}\,\mathcal{W}^2 + \frac{1}{8}\int {\rm d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{W}}^2 + \frac{1}{4}c\int {\rm d}^{12}\mathcal{Z}\,\ln\mathcal{W}\,\ln\bar{\mathcal{W}} + \cdots\,, \tag{3.30}$$

where the third term represents the leading quantum correction computed in [33, 24].

In our opinion, the perturbative scheme of solving the self-duality equation via the $\mathcal{N}=2$ Legendre transformation is difficult [24] as one has to invert the Legendre transformation. We suggest to look for the low-energy action of $\mathcal{N}=4$ SYM as a solution of the $\mathcal{N}=2$ duality equation (3.10). This equation is easy to deal with and it implies self-duality via Legendre transformation.

The low-energy effective action we are looking for should be in addition invariant under the $\mathcal{N}=2$ superconformal group. This means that, along with the structures given in (3.30), the action may involve the following manifestly superconformal functionals [30]

$$\mathcal{S}_1 = \int {\rm d}^{12}\mathcal{Z}\,\ln\mathcal{W}\,\Lambda(\nabla\ln\mathcal{W}) + {\rm c.c.}\,, \tag{3.31}$$

$$\mathcal{S}_2 = \int {\rm d}^{12}\mathcal{Z}\,\Upsilon(\nabla\ln\mathcal{W}\,,\,\bar\nabla\ln\bar{\mathcal{W}})\,, \tag{3.32}$$

where

$$\nabla \equiv \frac{1}{\bar{\mathcal{W}}^2}\,\mathcal{D}^4\,, \qquad \bar\nabla \equiv \frac{1}{\mathcal{W}^2}\,\bar{\mathcal{D}}^4\,, \tag{3.33}$$

and Λ and Υ are arbitrary holomorphic and real analytic functions, respectively. The superfields $\nabla\ln\mathcal{W}$ and $\bar\nabla\ln\bar{\mathcal{W}}$ prove to be superconformal scalars [30]. The main property of the operators (3.33) is that, for any superconformal scalar Ψ, $\nabla\Psi$ and $\bar\nabla\Psi$ are also superconformal scalars.

In components, the functionals (3.30), (3.31) and (3.32) contain all possible structures which involve the physical scalar fields $\varphi = \mathcal{W}|_{\theta=0}$ and the electromagnetic field strength F_{ab} (where $F_{\alpha\beta} \propto \mathcal{D}_\alpha{}^i\mathcal{D}_{\beta\,i}\mathcal{W}|_{\theta=0}$) without derivatives, along with terms containing derivatives and auxiliary fields. Simple power counting determines the necessary number of covariant derivatives in the action in order to produce a given power of F. Since $F \propto \mathcal{D}^2\mathcal{W}$, there should be $4n$ $\mathcal{D}$'s in the superfield lagrangian to get F^{4+2n} (additional 8 derivatives come from the superspace measure, $\int {\rm d}^{12}\mathcal{Z} = \int {\rm d}^4x\,\mathcal{D}^4\bar{\mathcal{D}}^4$).

We are looking for a perturbative solution of (3.10) in the framework of the momentum expansion or, equivalently, as a series in powers of ∇ and $\bar\nabla$. But with the Ansatz $\mathcal{S} = \mathcal{S}_{\rm lead} + \mathcal{S}_1 + \mathcal{S}_2$ it is easy to see that no solution of (3.10) exists. To obtain a consistent perturbation theory, we should allow for higher derivatives. More precisely, we should add new terms such that any number of operators ∇ and $\bar\nabla$ are inserted in the Taylor expansion of Υ (3.32). In other words, $\mathcal{S}_2$ should be extended

JHEP03(2000)034

to a more general functional $\hat{\mathcal{S}}_2$ which can be symbolically written as[8]

$$\hat{\mathcal{S}}_2 = \int \mathrm{d}^{12}\mathcal{Z}\, \hat{\Upsilon}(\nabla \ln \mathcal{W}\,,\; \bar{\nabla}\ln \bar{\mathcal{W}}\,,\; \nabla\,,\; \bar{\nabla})\,. \tag{3.34}$$

For the action

$$\mathcal{S}[\mathcal{W},\bar{\mathcal{W}}] = \mathcal{S}_{\rm lead} + \mathcal{S}_1 + \hat{\mathcal{S}}_2 \tag{3.35}$$

the equation of motion can be represented in terms of

$$\mathrm{i}\,\mathcal{M} \equiv 4\,\frac{\delta}{\delta\mathcal{W}}\,\mathcal{S}[\mathcal{W},\bar{\mathcal{W}}] = \mathcal{W}\Big\{1 + \bar{\nabla}\Gamma\Big\}\,, \tag{3.36}$$

for some functional $\Gamma(\ln\mathcal{W}, \ln\bar{\mathcal{W}}, \nabla, \bar{\nabla})$ such that $\Gamma = c\ln\bar{\mathcal{W}} + O(\nabla)$. Then, the duality equation (3.10) is equivalent to

$$\mathrm{Im}\int \mathrm{d}^{12}\mathcal{Z}\;\{2\,\Gamma + \Gamma\,\bar{\nabla}\,\Gamma\} = 0\,. \tag{3.37}$$

In the framework of perturbation theory, the procedure of solving of eq. (3.37) amounts to simple algebraic operations. To low order in the perturbation theory, the solution reads

$$\begin{aligned}
\mathcal{S} &= \frac{1}{8}\int \mathrm{d}^8\mathcal{Z}\,\mathcal{W}^2 + \frac{1}{8}\int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{W}}^2 + \frac{1}{4}\int \mathrm{d}^8\bar{\mathcal{Z}}\,\mathcal{L}\,,\\
\mathcal{L} &= c\,\ln\mathcal{W}\,\ln\bar{\mathcal{W}} + \frac{1}{4}\,c^2\Big(\ln\mathcal{W}\,\nabla\ln\mathcal{W} + \text{c.c.}\Big) + \\
&\quad + \frac{1}{4}\,c^3\,d\,(\nabla\ln\mathcal{W})\,\bar{\nabla}\ln\bar{\mathcal{W}} - \frac{1}{8}\,c^3\Big(\ln\mathcal{W}\,(\nabla\ln\mathcal{W})^2 + \text{c.c.}\Big) + \\
&\quad + \frac{1}{16}\,c^4\Big((1-4d)\,(\nabla\ln\mathcal{W})^2\,\bar{\nabla}\ln\bar{\mathcal{W}} + (2d-1)\,(\nabla\ln\mathcal{W})\,\bar{\nabla}\nabla\ln\mathcal{W} + \\
&\qquad\qquad + \frac{5}{3}\,\ln\mathcal{W}\,(\nabla\ln\mathcal{W})^3 + \text{c.c.}\Big) + O(\nabla^4)\,.
\end{aligned} \tag{3.38}$$

Here d is the first parameter in the derivative expansion of $\mathcal{S}$ which is not fixed by the $\mathcal{N}=2$ duality equation (3.10). Note that if we had only imposed the condition of self-duality under Legendre transformation, as was done in [24], we could not have fixed the coefficent $-\frac{1}{8}c^3$ of the fourth term in $\mathcal{L}$. In general, for any self-conjugate monomial in the expansion of $\mathcal{S}$, like $(\nabla\ln\mathcal{W})\,\bar{\nabla}\ln\bar{\mathcal{W}}$, the corresponding coefficient is not determined by eq. (3.10) in terms of those appearing in the structures in $\mathcal{S}$ with less derivatives. However, such coefficients can be fixed if one imposes some additional conditions on the solution of eq. (3.10). For example, one can require the solution to reduce to a given $\mathcal{N}=1$ action under the condition $\mathcal{W}| = \text{const}$.

[8] There exist more general superconformal invariants of the $\mathcal{N}=2$ vector multiplet [30], as compared to the action (3.34), and some of them were determined in [24] from the requirement of scale and $\mathrm{U}(1)_R$ invariance. It suffices for our purposes that (3.34) provides a consistent Ansatz to solve the $\mathcal{N}=2$ duality equation (3.10).

It should be pointed out that the c^3-corrections in (3.38) have been determined in [24] by solving the self-duality equation via the $\mathcal{N}=2$ Legendre transformation. To compute the $\mathcal{O}(c^4)$ term via the duality equation (3.10) involves only elementary algebraic manipulations.

As is seen from (3.38), solutions of the duality equation (3.10) contain higher derivative structures $\bar{\nabla}\nabla \ln \mathcal{W}$, $\nabla\bar{\nabla}\nabla \ln \mathcal{W}$, etc. What is the fate of such terms? The striking result of [24] is the fact that, to the order c^3, there exists a nonlinear $\mathcal{N}=1$ superfield redefinition which eliminates all higher derivative (accelerating) component structures (contained already in the first term of $\mathcal{L}$ (3.38)). The price for such a redefinition is that the original linear $\mathcal{N}=2$ supersymmetry turns into a nonlinear one being typical for D3-brane actions [34]. The nonlinear redefinition of [24] eliminates the higher derivative terms to some order of perturbation theory, but it in turn generates new such terms at higher orders in the momentum expansion. Therefore, in order for such a nonlinear redefinition to be consistently defined, the superfield action should involve higher derivatives of arbitrary order. The duality equation (3.10) might guarantee the existence of a consistent redefinition to eliminate acceleration terms.

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A. $\mathcal{N}=0$ duality invariant models

In this appendix we give several equivalent forms of the Gallard-Zumino equation (1.3) by representing the lagrangian $L(F_{ab})$ as a real function of one complex variable,

$$L(F_{ab}) = L(\mathcal{U},\bar{\mathcal{U}})\,, \qquad \mathcal{U} = \mathcal{F} + \mathrm{i}\,\mathcal{G}\,,$$
$$\mathcal{F} = \frac{1}{4}F^{ab}F_{ab}\,, \qquad \mathcal{G} = \frac{1}{4}F^{ab}\tilde{F}_{ab}\,. \tag{A.1}$$

The theory is parity invariant iff $L(\mathcal{U},\bar{\mathcal{U}}) = L(\bar{\mathcal{U}},\mathcal{U})$.

One calculates $\tilde{G}$ (1.2) to be

$$\tilde{G}_{ab} = \Big(F_{ab} + \mathrm{i}\,\tilde{F}_{ab}\Big)\frac{\partial L}{\partial \mathcal{U}} + \Big(F_{ab} - \mathrm{i}\,\tilde{F}_{ab}\Big)\frac{\partial L}{\partial \bar{\mathcal{U}}}\,, \tag{A.2}$$

and the Gallard-Zumino equation (1.3) takes the form

$$\mathrm{Im}\left\{\mathcal{U} - 4\mathcal{U}\left(\frac{\partial L}{\partial \mathcal{U}}\right)^2\right\} = 0\,, \tag{A.3}$$

which is equivalent to the equations obtained in [4, 7] but turns out to be more convenient for supersymmetric generalizations. If one splits L into the sum of Maxwell's part and an interaction,

$$L = -\frac{1}{2}\left(\mathcal{U} + \bar{\mathcal{U}}\right) + L_{\rm in}\,, \qquad L_{\rm in} = \mathcal{O}(|\mathcal{U}|^2)\,, \tag{A.4}$$

the above equation turns into

$$\mathrm{Im}\left\{\mathcal{U}\,\frac{\partial L_{\rm in}}{\partial \mathcal{U}} - \mathcal{U}\left(\frac{\partial L_{\rm in}}{\partial \mathcal{U}}\right)^2\right\} = 0\,. \tag{A.5}$$

We restrict $L_{\rm in}$ to be a real analytic function of $\mathcal{U}$ and $\bar{\mathcal{U}}$. Then, every solution of eq. (A.5) is of the form

$$L_{\rm in}(\mathcal{U},\bar{\mathcal{U}}) = \mathcal{U}\,\bar{\mathcal{U}}\;\Omega(\mathcal{U},\bar{\mathcal{U}})\,, \qquad \Omega = \mathcal{O}(1)\,, \tag{A.6}$$

where Ω satisfies

$$\mathrm{Im}\left\{\frac{\partial(\mathcal{U}\,\Omega)}{\partial \mathcal{U}} - \bar{\mathcal{U}}\left(\frac{\partial(\mathcal{U}\,\Omega)}{\partial \mathcal{U}}\right)^2\right\} = 0\,. \tag{A.7}$$

Note that for any solution $L_{\rm in}(\mathcal{U},\bar{\mathcal{U}})$ of (A.5), or any solution $\Omega(\mathcal{U},\bar{\mathcal{U}})$ of (A.7), the functions

$$\hat{L}_{\rm in}(\mathcal{U},\bar{\mathcal{U}}) = \frac{1}{\kappa^2}\,L_{\rm in}(\kappa^2\,\mathcal{U},\kappa^2\,\bar{\mathcal{U}})\,, \qquad \hat{\Omega}(\mathcal{U},\bar{\mathcal{U}}) = \kappa^2\,\Omega(\kappa^2\,\mathcal{U},\kappa^2\,\bar{\mathcal{U}}) \tag{A.8}$$

are also solutions of eqs. (A.5) and (A.7), respectively, for arbitrary real parameter κ^2.

Up to a trivial rescaling, eq. (A.7) coincides with the $\mathcal{N}=1$ duality equation (2.14). Therefore, any non-supersymmetric duality invariant model admits a $\mathcal{N}=1$ supersymmetric extension given by eqs. (2.10) and (2.14). It is easy to read off the bosonic sector of the action (2.10). For vanishing fermionic fields, $W_\alpha|_{\theta=0} = 0$, one finds [28]

$$\frac{1}{8}\,D^2\,W^2|_{\theta=0} = \mathcal{U} - 2\,\mathcal{D}^2\,, \tag{A.9}$$

where $\mathcal{D}(x)$ is the auxiliary field of the $\mathcal{N}=1$ vector multiplet. If we take the solution $\mathcal{D}=0$ of the equation of motion for $\mathcal{D}$, then the action (2.10) reduces to a generic $\mathcal{N}=0$ duality invariant model.

B. $\mathcal{N} = 2$ BI action and Legendre transformation

To prove that the system defined by eqs. (3.19) and (3.20) is self-dual under Legendre transformation, we replace the action (3.19) by the following one

$$\mathcal{S} = \frac{1}{4}\int \mathrm{d}^8\mathcal{Z}\left\{\mathcal{X}[\mathcal{W},\bar{\mathcal{W}}] - \mathrm{i}\,\mathcal{W}\mathcal{W}_{\mathrm{D}}\right\} + \frac{1}{4}\int \mathrm{d}^8\bar{\mathcal{Z}}\left\{\bar{\mathcal{X}}[\mathcal{W},\bar{\mathcal{W}}] + \mathrm{i}\,\bar{\mathcal{W}}\bar{\mathcal{W}}_{\mathrm{D}}\right\}, \qquad \text{(B.1)}$$

where $\mathcal{W}$ is now considered to be an unconstrained chiral superfield, and its dual chiral strength $\mathcal{W}_{\mathrm{D}}$ reads

$$\mathcal{W}_{\mathrm{D}} = \bar{\mathcal{D}}^4\mathcal{D}^{ij}\,U_{ij}\,, \qquad \text{(B.2)}$$

with U_{ij} an unconstrained real prepotential. The equation of motion for U_{ij} implies the Bianchi identity (3.5), and hence the action reduces to (3.19). On the other hand, varying the action with respect to $\mathcal{W}$ leads to

$$\mathcal{W}_{\mathrm{D}} = \mathcal{M}\,, \qquad \text{(B.3)}$$

where $\mathcal{M}$ is given in eq. (3.24). The latter equation can be solved to express $\mathcal{W}$ in terms of $\mathcal{W}_{\mathrm{D}}$ and its conjugate. Instead of doing this explicitly, we note that eqs. (3.28) and (B.3) allow one to rewrite the action (B.1) as

$$\mathcal{S} = \frac{1}{4}\int \mathrm{d}^8\mathcal{Z}\,\mathcal{X}_{\mathrm{D}} + \frac{1}{4}\int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{X}}_{\mathrm{D}}\,, \qquad \text{(B.4)}$$

where

$$\mathcal{X}_{\mathrm{D}} \equiv -\frac{1}{1-\mathcal{Q}}\,\frac{\mathcal{X}}{1-\bar{\mathcal{D}}^4\bar{\mathcal{X}}} - \mathcal{P}\,\frac{1}{1-\bar{\mathcal{Q}}}\,\frac{\bar{\mathcal{X}}}{1-\mathcal{D}^4\mathcal{X}}\,. \qquad \text{(B.5)}$$

Using eqs. (3.28) and (B.3) once more, one can prove that $\mathcal{X}_{\mathrm{D}}$ satisfies the constraint

$$\mathcal{X}_{\mathrm{D}} = \mathcal{X}_{\mathrm{D}}\,\bar{\mathcal{D}}^4\bar{\mathcal{X}}_{\mathrm{D}} + \frac{1}{2}\,\mathcal{W}_{\mathrm{D}}{}^2\,. \qquad \text{(B.6)}$$

This completes the proof.

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